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ABSTRACT

This paper speaks to the challenges, risks, and rewards of teaching about linear functions in a technology-rich environment from a constructivist perspective by describing a Foundations of Algebra class, which was designed and taught to junior high school students in such an environment in the summer of 1991. The design and implementation of the class were based on cognitive science research on linear functions and made use of computer software designed to help students make connections between the algebraic and graphical representations of functions. The introduction to this paper describes the Functions Research Group at the University of California at Berkeley and its research on how students learn about linear functions using the computer program "Grapher." Technology and constructivism are then briefly discussed and the course is described in detail, noting that it met for a total of 36 hours and the students were 22 racially and socioeconomically diverse adolescents in the eighth and ninth grades. A discussion of challenges posed by the constructivist approach in the classroom presents two vignettes that consider the teacher's role in two types of activities: computer labs and activities where students worked in small groups to explore, extend, and apply ideas, and whole class discussions where the results of these explorations were probed and extended. Some rewards of constructivism in the classroom are suggested, and a discussion of issues involved in constructivist teaching concludes the paper. (Contains 12 references.) (ALF)

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From the Laboratory to the Classroom:

A Technology-Intensive Curriculum for Functions and Graphs

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Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA, April 20-24, 1992, as part of the symposium "Understanding, tutoring, and teaching a complex subject matter domain: Linear equations and their graphs in the Cartesian plane."

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Abstract

This paper speaks to the challenges, risks, and rewards of teaching about linear functions in a technology-rich environment from a constructivist perspective. I describe a Foundations of Algebra class I designed and taught to junior high school students in the summer of 1991. A primary theme of the course was linear functions – their tabular, algebraic, and graphical representations, their abstract as well as real world incarnations. The design and implementation of the class were based on cognitive science research on linear functions and made use of computer software designed to help students make connections between the algebraic and graphical representations of functions. This paper presents and discusses some entailments of constructivism in a technology-rich classroom.

Introduction

Since 1985, the Functions Research Group at The University of California, Berkeley has been conducting research on how students learn about linear functions (e.g., Meira, 1991; Moschkovich, 1992; Schoenfeld, Smith, and Arcavi, in press). The centerpiece of this research is a computer program called GRAPHER that allows students to manipulate the parameters in equations and to observe the graphical entailments of these manipulations (Schoenfeld, 1990). For the past three and a half years I have used this research to design instructional activities to help students explore and make sense of linear functions. The activities have been evaluated through microanalysis of pairs of students working on them in a laboratory setting (Magidson, 1991). Last summer I had the opportunity to design and teach a Foundations of Algebra course. One of the main themes of the course was linear functions, so I worked to adapt these activities for a classroom setting. This paper gives an overview of the resulting pilot study and speaks to the complexities of teaching through a constructivist lens.

This paper can be seen as a companion paper to Arcavi & Schoenfeld (1992). Arcavi and Schoenfeld explore the complexities of tutoring through a constructivist lens, noting the subtle and complex decision-making issues a tutor faces when trying to co-construct knowledge with just one student. Co-construction with an entire classroom of students ups the ante considerably. I discuss the trade-offs, difficulties, and benefits of pursuing constructivist goals in the complex environment of the classroom.

Technology and constructivism

The title of this paper promises a description of a technology-rich curriculum, yet the focus of the paper is constructivism. This is not accidental. The paper *is* about technology in the classroom, but it is difficult to separate entailments of the technology from entailments that emerge from a sense-making, exploratory, problem-driven curriculum: each influences and informs the other. In the context of constructivism, the technology provided an exploratory environment that served as a centerpiece for the curriculum. The technology itself did not dictate a sense-making environment, but merely facilitated it.[1] Thus in order to talk about entailments of a technology-rich environment, one needs to talk explicitly about one's epistemological assumptions and how they play out in the classroom.

There are many brands of constructivism. I use the term to mean (à la Kilpatrick, 1987) that learners learn by building on what they know in ways that make sense to them; that in order for someone to learn something, he or

[1]Had the technology consisted of drill and practice, sensemaking would have been considerably more difficult, though not impossible. Similarly, a technological microworld or tool does not guarantee that sensemaking will occur: it is certainly possible to do rote work with Grapher or Logo or any other computer tool intended for exploration.

she must actively process that knowledge by relating it to previous knowledge. I assume that there is a body of core mathematics we want students to explore, and that we can create learning situations that will help them to do so. I also assume that learners will react to whatever sort of learning situation we provide by trying to relate it to what they know.

The course

The course, entitled "Foundations of Algebra," was offered in the summer of 1991 as part of U. C. Berkeley's Academic Talent Development Program, a summer program for highly motivated students from the greater San Francisco Bay Area. The course met for three-hour sessions twice weekly for six weeks, resulting in a total of 36 class hours or the equivalent of 7-10 weeks of a typical school year. The purpose of the class was to help students build a conceptual base from which they could then approach algebra with greater understanding. The class focused on two themes central to an understanding of elementary algebra (and much of higher mathematics): function and abstraction.[2] Approximately half of each class period was devoted to explorations of each theme, though there were of course many overlaps between the two. In accord with constructivist principles, the class was designed to provide an environment in which students were supported and encouraged to explore and make sense of mathematical ideas. Thus the design incorporated both content goals (linear functions and beginning abstraction) and epistemological goals (encouragement and support for student constructions).

The design of the class was based on cognitive research on linear functions (Schoenfeld, Smith, and Arcavi, in press; Moschkovich, Schoenfeld, and Arcavi, in press) which lays out the complexity of the domain and the importance of building connections across the representations of tables, equations, and graphs. Moschkovich, Schoenfeld, and Arcavi write, "developing competency with linear relations means learning which perspectives and representations can be profitably employed in which contexts, and being able to select and move fluently among them to achieve one's desired ends." (p. 5)

The spirit and goals of the class were in line with the current NCTM Standards which speak to the importance of the concept of function. According to the Standards, high school students should be able to do the following: "represent and analyze relationships using tables, verbal rules, equations, and graphs; translate among tabular, symbolic, and graphical representations of functions; recognize that a variety of problem situations can be modeled by the same type of function; and analyze the effects of parameter changes on the graphs of functions." (p.154.) Further, instructional

[2]By "abstraction", I mean motivating a need for an abstract representation of a situation (symbolic or graphical): seeing how constructing such representations enhances one's understanding or ability to manipulate a situation.

methods are to focus on investigation, sense-making, meaning construction and flexibility, and should include small group work, explorations, and whole class discussions where the teacher serves as moderator. Technology, in particular, serves to transform the mathematics classroom into a science laboratory where students "investigate, conjecture, and verify their findings." (NCTM, 1989.)

The purpose of the computer labs that took place during class time was to give students the opportunity to explore aspects of linear functions interactively. Computer labs consisted of structured guided discovery, open-ended problems, and games which required students to grapple with various aspects of the domain. In the starburst problem shown in Figure 1 (Magidson, in preparation), for example, students explore qualitative aspects of slope – slope as a measure of steepness, the symmetry of positive and negative slopes, benchmarks slopes of 0, ±1, ±∞, and issues of spacing. Students later unpack the mechanism behind slope (ratio of directed line segments) when they work with computer-generated staircases. They then return to the starburst problem to produce a precise (rather than approximate) solution. Replicating pictures of parallel lines gives them the opportunity to explore y-intercept.

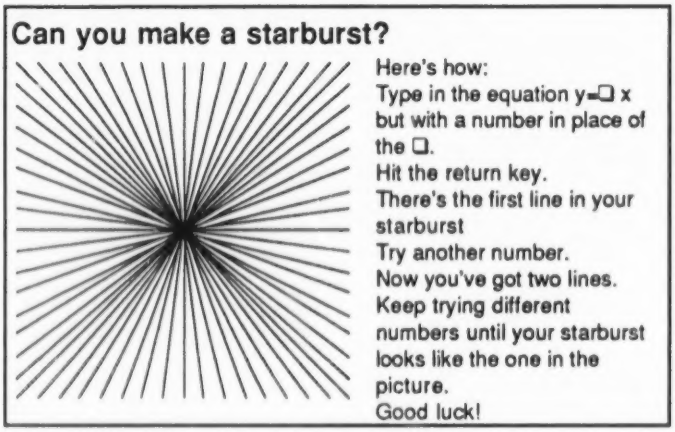
Can you make a starburst?
Here's how:
Type in the equation y=❑ x but with a number in place of the ❑.
Hit the return key.
There's the first line in your starburst
Try another number.
Now you've got two lines.
Keep trying different numbers until your starburst looks like the one in the picture.
Good luck!

Figure 1. The Starburst Problem

Tasks such as line matching and Black Blobs[3] (Figure 2) help students to refine their knowledge of slope and intercept and their independence

[3]Line matching involves replicating a given line by finding its equation. Black Blobs (inspired by Green Globs, Dugdale, 1982) is a computer game where students are given a grid with thirteen random blobs (squares with sides of unit length one). Their goal is to pop the blobs by typing equations whose graphs run through them. Scoring is exponential based on the number of blobs popped in a single shot: one blob yields 2 points; two blobs yield 2+4=6 points, three blobs yield 2+4+8=14 points, and so on. See Figure 2 for a sample screen.

(changing slope rotates a line around the y-intercept; changing y-intercept translates the line along the y-axis) as they attempt to produce strategically placed lines in a game setting.

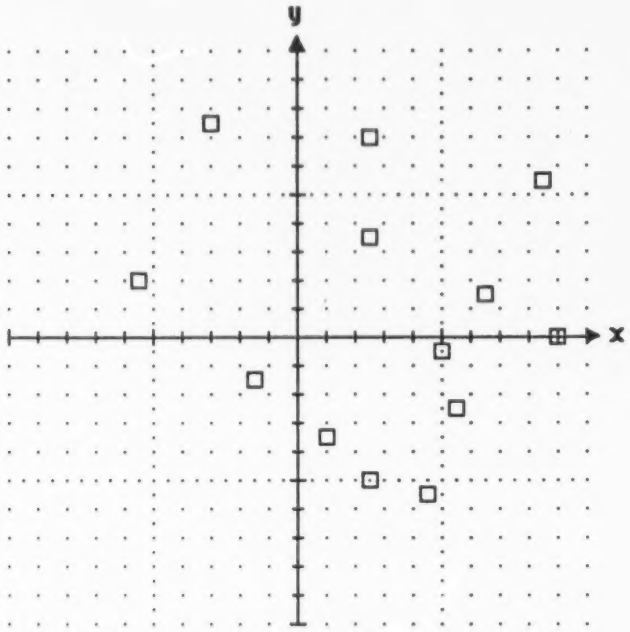

Figure 2. A Black Blobs Screen

The students

The class consisted of 22 racially and socioeconomically diverse junior high school students about to enter eighth and ninth grade. Almost all of the students had taken a pre-algebra course the year before and were planning to take Algebra I the following fall. Within these constraints, there was a wide range of mathematical knowledge, sophistication, and confidence: two thirds of the students considered themselves average mathematics students; one third considered themselves strong mathematics students. The point here is that these students were not as atypical as one might assume given the context of the course. Almost all of them were highly motivated since they (or their parents) chose that they spend six weeks of their summer studying mathematics, but their previous experiences and successes with math were diverse.

Data collection

All full class discussions were video- and audio-taped. The recorders were kept running during group activities, but no attempts were made to move the camera or microphone closer to a particular group of students since this was thought to be too disruptive to the activity. During each computer

lab video cameras were set up behind two pairs of students; since students switched partners for each lab, different pairs were taped each time. Other data include photocopies of all overhead transparencies (I used the overhead projector as a chalkboard), student exams, and student assignments[4].

Challenges of Constructivism in the Classroom

As described earlier, class time was focused on two main types of activities: a) computer labs and other activities where students worked together in small groups to explore, extend, and apply ideas; and b) whole class discussions where the results of these explorations were probed and extended. The following two sections discuss the teacher's role within these activities from a constructivist perspective.

Vignette #1: Distributed Constructions

Arcavi and Schoenfeld (1992) demonstrate the difficulties a tutor faces if she takes constructivism seriously: she must understand and evaluate the student's constructions and then make decisions about how best to help the student proceed, doing justice both to the mathematical validity of the student's ideas and the student's capacity to pursue these ideas to fruition. All of this must be done on the spot, in real time. These constraints pose a substantial challenge to the tutor. Compounded with the additional constraints of a classroom, however, the complexity and challenge of constructivism in action increases many times over. It is important to realize that a tutor shaping an intervention has two significant advantages over a classroom teacher. First, she is present during the entire session. Second, her decisions are restricted to the needs of a single student. During classroom explorations, in contrast, a teacher is usually circulating throughout the room trying to monitor the activity of each group or pair. If she does sense the need for an intervention, she has only a few minutes at most to assess the situation and interact with the students. Her intervention is thus based on very limited data and must be short and to the point: there is no time for a prolonged interaction with a single pair of students[5] since others will also need her attention. Further, her decision must take into account not only the needs of the pair, but how those needs fit with the larger goals of the class.

What does constructivist teaching look like within the context of distributed explorations? Practically speaking, the teacher cannot give one pair her undivided attention. She can, however, help students to clarify goals

[4]With the exception of occasional textbook exercises.

[5]She does have other options, not available in a lab setting, such as helping students to help each other. One common ploy of mine, when I notice that two groups have arrived at contrasting solutions to the same problem is to set them to work trying to understand each other's thinking to resolve the conflict. The students may need help structuring their discussion or deciding what makes for a convincing mathematical argument, but these are valuable skills to pursue; further, the intervention touches twice as many students.

and assumptions, ask provocative questions, help students to collaborate productively, and help students stay on task. These are all common interventive activities that teachers learn to do, but their entailments illustrate some of the tensions involved when constructivism is taken seriously in a classroom context.

I use the following vignette to discuss the entailments of a very minor intervention. The setting was a computer lab where the students were to work on two activities in the context of a computer game called Black Blobs.[6] First, students were to play one game to completion while aiming for the highest possible score. Second, students were to experiment with alternate forms of linear equations (slope/y-intercept, slope-x/intercept, point/slope) in the context of Black Blobs and to decide which forms were most useful in which game context (they were then to write advice for a friend playing the game).

I'd been keeping an eye on two students who were still playing their first Black Blobs game after the other students had moved on to the second activity. Their strategies did not seem terribly productive – they seemed to be having limited success in producing the lines they desired and did not appear to be making much progress. The period was running out and I was concerned that this pair have the opportunity to work on the second half of the assignment before the end of the computer lab since we would be discussing it later in class, so I asked the students to move on to the second activity.

This was not a remarkable intervention. On the contrary, it was commonplace. Of interest are its entailments, because it can be viewed as working both for and against the constructions of the two students involved. One perspective is that my intervention came from the outside: I made the decision to intervene without taking the time to fully understand what the students were working on, and, as such, was not respectful of and failed to help them build on their current constructions. On the other hand, I knew that the second activity provided the basis for the classroom discussion to follow. I wanted these students both to be able to contribute to and take from the discussion – to construct their own meaning from other students' discoveries. They had little chance of doing so if they hadn't at least experimented with the second activity. I may have interfered with one set of constructions, but my goal was to help set the stage for another.

This intervention was of note primarily because it happened to be videotaped, and I had the opportunity to look at the tape many months later. Even with my full attention to devote to the videotape, much was unclear. The students' goals were often unspoken as was their reasoning. Had I been sitting behind them during the entire class watching intently (but not intervening), I still might not have been sure what they were up to.

[6]See Figure 2 and corresponding footnote for a description of Black Blobs.

However, after two viewings, it was clear to me that the students' goals and strategies changed and improved during the course of the session: they became more skillful at the game during the computer lab. By intervening, I had interrupted some worthwhile mathematical activity.

Knowing this does not help me to pass judgment on the worth of the intervention. There is no way to know what use the students would otherwise have made of that time, and whether their exploration would have continued to be productive. What they got out of my intervention was ten minutes of exposure to the slope/x-intercept form of the equation -- enough time for them to plan a shot, type an incorrect equation (they'd used the y- rather than the x-intercept), recognize the problem, correct it, and then adjust their estimation slightly to shoot the desired targets. I can hope that this ten-minute intervention gave them a better chance of participating in the class discussion that ensued, but I cannot know how that time would have been best spent.

What I take from this situation is that it can be hard to recognize learning when it happens, especially in the classroom. Students are not always clear about their goals or motivations, but this does not necessarily mean that they are not learning. Further, even when a teacher <u>does</u> recognize learning in a distributed learning situation, her decision is not an easy one: she must weigh her goals for the class against her goals for individual students and their goals for themselves. In the vignette above, even if I had recognized the value of my students' activity, I would still have had to weigh the usefulness of the two alternatives for the pair involved as well as for its contribution to the class as a whole. Both can be seen as in the best interest of constructivism, and, as such, are important to consider.

Vignette #2: Class Constructions

The preceding section illustrated some of the issues teachers face when trying to support student constructions during exploratory activities. This section speaks to the role of full class discussions and argues for the need for class constructions as well as individual constructions.

The starburst problem (shown in Figure 1) is one I've been thinking and writing about for several years. It provides a structure in which students can explore the properties of slope in an open-ended way because the goal is well-specified and the method is not. Laboratory and classroom studies (Magidson, in preparation) show a wide range of student constructions within the starburst problem. Some constructions are variations on a theme[7]; others address distinct aspects of slope's behavior. Further, different students construct different subsets of these and combine them in different ways. (See Magidson, in preparation, for a more complete account.)

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[7] For example, "larger numbers produce steeper lines" and "to get a line between two lines, choose a slope number between the two numbers" are both instances of ordering.

After students work through the starburst problem, their ideas about slope are fragmented and incomplete. They have questions about their experiences and about what they observe on their classmates' screens (e.g. "How did they manage to space the sides so perfectly?" "How did they manage to get a horizontal line?"). If students were to pool their constructions, they would find that collectively they know a great deal about the domain, but that their individual discoveries are limited. I believe that this is a situation where a full class discussion is essential. Pursuing and contesting their constructions in a public setting can help students strengthen some ideas, eliminate others, and put together an underlying structure to link them. A class discussion gives students the opportunity to share and consolidate ideas, and is important for their individual understandings, but also helps them develop what I would refer to as "class constructions"[8] -- group understandings and vocabulary that can be used by the class to make sense of future activities and problems.

What are the constructivist entailments of full class discussions? One obvious drawback is that it is not possible to pursue all student constructions in a deep way: teachers are limited both by time and by the patience and attention of the class as a whole, and this will undoubtedly involve bypassing some student constructions. Further, some constructions are more worthy of pursuit in a full class setting because of their accessibility or where they may lead. Teachers will have to make choices (on the spot), not only about the value to the individual student of pursuing her construction, but of the value to the class as a whole. On the other hand, class discussions provide a forum in which students can articulate, test, and refine their constructions and see how they fit into a larger structure. Even if a student's own idea is not pursued, the student can learn how it might be pursued during the joint pursuit of other ideas.

Facilitating a full class discussion of student constructions is challenging, even when the task is as tightly constrained as the starburst problem. It involves the following activities: helping students articulate their ideas; helping other students understand, respond to, and challenge them; helping students decide when something is true; helping students to see connections between ideas; and helping student organize them into a larger whole. For the starburst problem, this involves working from a long list of fragmented observations about the domain to integrate them into a continuum of slopes ordered from $-\infty$ to ∞ (see Figure 3).

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[8] I do not mean to imply by this that all of the students in the class construct the same knowledge, but rather that the knowledge is constructed in a social setting and is shared by the participants. The students develop a common vocabulary and reference examples that can be used later to argue a point. Individual students may have access to pieces of that knowledge but the information is distributed throughout the class and no one, including the teacher, has access to all of it.

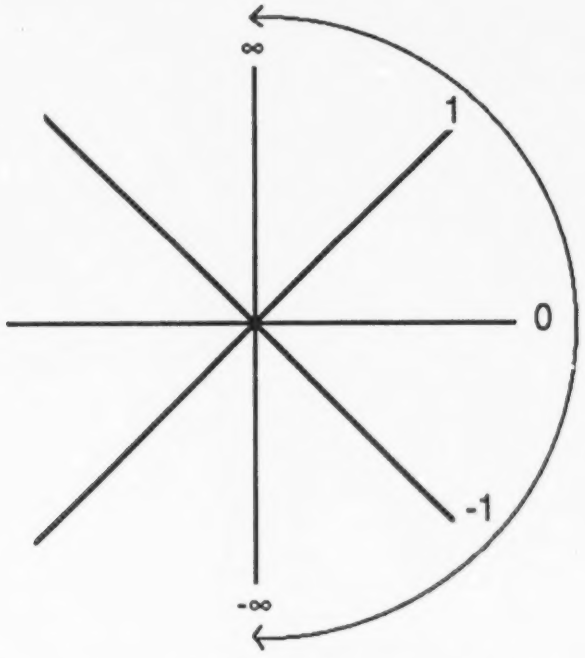

Figure 3: Slope as a continuum

The point here is not that individual student constructions are subsumed by the larger structure, but rather that they are connected to it. Individual constructions retain their validity, but can also be seen as part of a larger integrated structure. Leading the starburst discussion in the past has been a challenge. Last summer, I unintentionally upped the ante and the discussion bordered on overwhelming.

In the past when I've used this task, I've instructed students to use equations of the form $y = mx$ and to experiment by substituting different numbers for the m. Last summer, I decided to extend the task by asking students to first discover the characteristics of equations that produce lines through the origin. These students had worked extensively with generating tables of points that satisfied a given equation, plotting the points, and generalizing the line. I wanted them to connect those experiences to the starburst task, to discover for themselves what types of equations produce lines through the origin. Thus I changed the directions to read as follows:

> The starburst is made up of lines. See if you can find an equation that produces a line that is part of the Starburst. When you find one, draw that line on the recording sheet and write down the equation that produced it. Keep trying different equations until you have created a Starburst that looks like the one in the picture.

I did append the following footnote cautioning students to use equations that begin with the form y = : "Note that this computer program requires that your equation must begin with x = or y =. For the purpose of this activity, please restrict yourself to equations that begin y = ..." I did so because my agenda was slope, and equations of the form x = ny would deal with slope's reciprocal. What I failed to anticipate was not only that students would ignore the footnote, but that they would also invent two alternate forms of the equation that I, blinded by many years of working with the slope-intercept form, hadn't allowed for: y = x/n and x = y/m. Suddenly I found students using four perfectly valid forms of the equation. This multiplied by four my task of facilitating consolidation and synthesis. More importantly, it made it virtually impossible to generalize about the effect of the slope numbers on the resulting lines since, depending on the form used, the same number could produce a different line (e.g. students using a 3 could either end up with y = 3x or y = x/3).

It is important to note a distinction between tutoring and teaching. Had this situation evolved in a tutoring situation it would not have been particularly problematic because the tutor would have been present during its evolution and would have either chosen to intervene early on or decided to run with whichever form of the equation the student chose. The difficulty arose not from any one pair's problem solving but from the (classroom) need to eventually converge on some common interpretations and the diversity of problem spaces that emerged.

I did attempt some emergency rescue along the way. During the computer lab itself, I tried to find students who were using equations that began with x = and draw their attention to the footnote. Later, when I caught instances of y = x/m, I acknowledged students' successes but asked them to try to work with y = mx so that we'd be able to talk about their experiences with the rest of the class. Back in the classroom, we discussed the different forms and the need (and procedure) for putting them in a common form so that we could all discuss the same numbers. Even though students dutifully used their calculators to find the reciprocals of the numbers they had originally used, the original numbers, the ones that figured in their explorations, remained dominant.

From a constructivist point of view, my actions may appear suspect. After all, wasn't I actively denying student constructions for a seemingly arbitrary convention? Oughtn't I, as a good constructivist, to have scrapped my lesson plans and worked with what the students constructed? Perhaps. In my own defense, I knew from previous classes' discussions about starburst and from the data I'd analyzed (Magidson, in preparation) that the discussion was likely to be quite complex even with the constraint that all equations be of the form y = mx; it threatened to be overwhelming with four competing forms of the equation. Further, the class needed to construct some common knowledge about starburst if we were to talk about slope eventually. Yet, to be honest, I note that my emergency rescue attempts were of marginal value --

the student constructions were so powerful that the resulting discussion was unusually confusing and unsatisfying.

Wh. ' can we learn from this? For myself as a teacher, I now know that asking students to first discover appropriate equations opens up a large field of possibilities that diverges from the goal of exploring slope. If I want students to discover the form of the equation on their own, I need to structure a separate exploration followed by a class discussion in which we discuss all four forms of the equation, explore their relationships (discovering that they are equivalent with proper algebraic manipulations), and decide to adopt a common one for the subsequent activity. From the point of view of classroom constructivism, two lessons emerge. First, in order for a class to work as a whole, there is a need for class activities and constructions so that students can build a common vocabulary for core mathematical ideas. The alternative seems a form of instructional anarchy in which each student pursues ideas of value to himself and thus it is extremely difficult for students work together or to learn in a whole group setting. Second, facilitating class discussions is extremely difficult even when the task is tightly constrained, because students exploring a foreign domain will find many different phenomena to be of interest.

Rewards of Constructivism in the Classroom

As these two vignettes demonstrate, constructivist teaching is hard work. In particular, the demands of co-constructing knowledge with a class full of students are exponentially greater than those of working with an individual student, and those demands are substantial (Arcavi & Schoenfeld, 1992). Teachers face constant intervention decisions which they must base on minimal information and analysis. Seemingly minor changes in an activity may have catastrophic results. By opening class discussion to student constructions, teachers set themselves up to take on the unexpected in a public context, again and again. This type of teaching is much more perilous and exhausting than traditional lecturing. Why would a teacher voluntarily take on such a job? In this section, I'd like to address some potential rewards of teaching through a constructivist lens.

First, last summer's students demonstrated a superior understanding of linear functions. I've taught about linear functions in the context of Algebra I many times. To my surprise, the Foundations of Algebra students found trivial exams that would have overwhelmed former algebra students (80% of the grades on the final exam were at least 90% correct).[9,10]

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[9]The exams contained both traditional textbook linear functions questions (e.g. "given a graph, write the equation"; "given an equation, draw the graph" "given two points, give the slope of the line that passes through them"; etc.) as well as explanatory questions ("why is it impossible to calculate the slope of a vertical line?" "Is it possible for a line with a

In addition, videotapes and photocopies of student assignments demonstrate that students were making connections and discoveries and posing questions beyond the realm of a standard algebra one treatment of function. For example, as part of students' nightly homework assignment, they were to write a brief summary of the ideas discussed in class and then to pose one or two questions -- either on topics of confusion or on topics they'd like to know more about. Following are a few examples of questions students asked, with my comments following in italics:

- Why isn't a line with a slope of 2 twice as steep (i.e. twice the angle) of a line with a slope of 1? *[I interpret this question as an example of constructivism in action -- this student is trying to reconcile her previous knowledge of angle with her new knowledge of slope.]*

- If you were to do run over rise would that give you the inverse of the slope? *[This student is trying to relate his new knowledge of slope to his old knowledge of inverse.]*

- Does the origin of the Cartesian plane (place where two axes meet) signify the exact middle of the whole plane? *[Here the student is trying to explore the ramifications of standard definitions.]*

- Why does it work to divide the rise by the run to get the slope? *[This student is trying to understand the mechanism of slope -- we'd discovered that slope can be found by dividing the rise by the run -- he wants to know why.]*

- Why is the slope of a vertical line undefined?

- Where does the name y-intercept come from? Why isn't it the x-intercept instead? Why did they pick on the y-axis? *[Here is a student who is trying to understand a convention that doesn't appear to make sense. Most lines have both an x-intercept and y-intercept -- why do we focus on one rather than the other?]*

- What is the difference between functions and equations?

- Is there a limit to the number of staircases a line can have?

- If you have a rise and run of 2/4, can you reduce it so that it's easier to find the slope? *[Again, an example of constructivism at work. Do the rules*

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negative slope to be steeper than a line with a positive slope? Explain." "Do all lines have a y-intercept? Why or why not?"

[10]I am not purporting a statistical comparison -- this was a pilot course very different in nature from a standard Algebra I course; no attempt was made at a strict comparison. When I speak of relative difficulty, I speak fro my experience as a teacher and as a researcher.

for reducing fractions apply to slope? Is slope just a regular fraction or does it have different entailments?]

When we teach by lecturing, questions like these are not often readily apparent. This does not necessarily imply that students are not questioning conventions and contradictions[11], but simply that they are not doing so publicly. By actively seeking students' questions, teachers can get closer to the students' conception of the domain (as opposed to their own), and help them to address their questions directly.

Finally, I was initially concerned by the diversity of mathematical knowledge and sophistication as demonstrated on placement tests and in early problem solving activities -- in a traditional classroom, the wide range seemed a formula for disaster. Instead, within the structure of the class and the problems, there was room for a wide variety of approaches and levels of meaning. Students were able to choose meaningful (to them) paths to pursue during the group activities and then were able to share those discoveries during full class discussions. Further, by working together during group explorations, they were able to help each other sort out confusions and take some of the teaching burden off the (official) teacher.

Discussion

If we believe that people learn by active knowledge construction, we need to take this into account in our mathematics classrooms. Theoretically it makes sense for teachers to respect students' constructions, to try to understand them as fully as possible, and to help students build upon them. Practically, this makes sense as well, providing one has reasonable expectations about what is possible and appropriate for a classroom. As Arcavi and Schoenfeld (1992) demonstrate, constructivism is extremely difficult in a tutoring situation if one's goal is to fully understand and build on a student's constructions; further, pursuing a student's constructions is not always the best choice since some constructions are not fruitful and others may lead beyond the student's capacity to follow. In a classroom these issues are compounded by the sheer number of students (and thus constructions) one must attend to. The teacher is faced with less data per student, more information overall, and more issues to weigh when making decisions. In the face of this seemingly insurmountable task, it is easy to write off constructivism as untenable in a classroom, given the vast increase in complexity.

If we define "constructivist teaching" solely in terms of the individual learner and then try to apply it in the classroom we set ourselves an overwhelmingly difficult task. Constructivism in the classroom may well be impossible if we define it with the condition that teachers pursue each

[11] Indeed, constructivism suggests that they must be doing so in order to learn.

student's constructions to fruition, that teachers fully understand each student's constructions before acting. On the other hand, if we are willing to accept the imprecision and limitations imposed by the classroom I believe it is possible to teach mathematics effectively through a constructivist lens. We can create situations that generally support and encourage student constructions. The paradox is that in order to do so generally, we can't always honor individual constructions. Reasonable goals include the following: to pursue some students' constructions in a full class setting in order to model sense-making activity; to provide a support structure in which student constructions are valued and encouraged; and to provide a class structure that has room for different levels of construction and understanding within shared core ideas.

Whether viewed as an art or a science, teaching is imprecise work, especially when one attempts to help students build on their ideas. Any teacher who teaches through a constructivist lens will (at least occasionally) misinterpret student constructions or cut off worthwhile mathematical activity or choose an unproductive path to pursue, but there is room for these mistakes in the larger context of a constructivist classroom. By providing an environment where sense-making is valued and pursued, we can still help students to take their ideas seriously and to learn to explore mathematics more rigorously. In doing so, we can reasonably support constructivism in the classroom.

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